Bound states in the transfer matrix spectrum for general lattice ferromagnetic spin systems at high temperature

Ricardo S. Schor and Michael O'Carroll*
Departamento de Física ICEx, Universidade Federal de Minas Gerais, Belo Horizonte, Minas Gerais, Brazil
(Received 7 July 1999)

We obtain different properties of general d dimensional lattice ferromagnetic spin systems with nearest neighbor interactions in the high temperature region ($\beta \ll 1$). Each model is characterized by a single site *a priori* spin distribution, taken to be even. We state our results in terms of the parameter $\alpha = \langle s^4 \rangle - 3\langle s^2 \rangle^2$ where $\langle s^k \rangle$ denotes the kth moment of the *a priori* distribution. Associated with the model is a lattice quantum field theory that is known to contain particles. We show that for $\alpha > 0$, β small, there exists a bound state with mass below the two-particle threshold. For $\alpha < 0$, bound states do not exist. The existence of the bound state has implications on the decay of correlations, i.e., the four-point function decays at a slower rate than twice that of the two-point function. These results are obtained using a lattice version of the Bethe-Salpeter equation in the ladder approximation. The existence and nonexistence results generalize to N-component models with rotationally invariant *a priori* spin distributions.

PACS number(s): 75.10.Hk, 11.10.St

I. INTRODUCTION AND RESULTS

In this work we obtain different properties of general d dimensional lattice ferromagnetic classical spin systems with nearest neighbor interactions in the high temperature region. Each such system is characterized by a single site *a priori* spin probability distribution. Associated with these systems is a lattice quantum field theory with Hamiltonian energy and field momentum operators living on a $(d-1)$-dimensional sublattice. The Hamiltonian is minus the logarithm of the transfer matrix [1,2]. The unusual properties are uncovered by a detailed study of the interaction of the particles of this underlying field theory. The idea of studying these systems via the transfer matrix is not new but up until now it has only been established that the low-lying energy-momentum (EM) spectrum consists of a particle with an isolated dispersion curve. These results imply exponential decay of correlation functions (CF) and the Ornstein-Zernicke behavior of the two-point CF [3,4]. Our results go beyond giving this information on the spectrum up to the two-particle threshold and have consequences for the decay of CF's.

Our basic result can be stated in terms of the sign of the quantity $\alpha \equiv \langle s^4 \rangle - 3\langle s^2 \rangle^2$, where the brackets are moments of the *a priori* distribution, taken to be even. We show that if $\alpha > 0$, the dominate interaction (which is local) is attractive and a bound state exists such as when there is an energy spectrum below the two-particle threshold. The mass of the bound state, denoted by m_b, is given by $m_b = 2m + \ln(1\text{-}\gamma) + 0(\beta)$, where $\gamma = \alpha/(2+2\langle s^2 \rangle^2)$ so that $0<\gamma<1$. For $\alpha<0$, the interaction is repulsive and there are no bound states. In the Gaussian case that corresponds to $\alpha=0$, the particles do not interact. The presence of bound states in the spectrum implies decay properties of CF's; for example, the four-point function has a slower than two-particle decay rate.

The spectral results established here are obtained using a lattice version of the Bethe-Salpeter (BS) equation (which employs a differently devised set of coordinates suitable for the lattice two-body bound state problem) in the ladder approximation. This approximation has proved to be quite reliable in cases where a complete mathematical treatment can be carried out [5,6].

We point out that for a wide class of models, CF inequalities have been established in [1,7], called Gaussian domination inequalities. The correlation inequalities imply spectral results, namely, the absence of bound states below the two-particle threshold. For small β, these results are included in our analysis since these models correspond to $\alpha<0$. In addition, our bound state results generalize to N-component spin models (see below), while Gaussian domination inequalities have only proved to hold for the scalar and Abelian ($N=2$) cases.

Our results on the existence of bound states generalize to N-component spin models where $s_i(x)$ is the ith component of $s(x) \in R^N$. The single spin distribution (SSD) is taken to be even and rotationally invariant. α, the parameter for the scalar spin case, is replaced by

$$\alpha_N = \langle (s\cdot s)^2 \rangle - \frac{N+2}{N}\langle s\cdot s \rangle^2,$$

and bound states exist for $\alpha_N>0$ and are excluded for $\alpha_N<0$. $\alpha_N=0$ corresponds to the Gaussian case.

We now turn to a more precise description of the class of models we treat and show how our results are obtained. For simplicity, we only consider explicitly the scalar spin case, but at the end we give the generalization to the N-component case. We let $s(x) \in R$, $x=(x_o,\vec{x}) \in Z^d$ denote the spin variable at the lattice site x. Formally, the generating function is $Z(J) = \int e^{(J,s)} e^{S(s)} d\mu(s)$, where $(J,s) = \Sigma_x J(x)s(x)$ and the interacting action is, with $\beta>0$ and small,

$$S(s) = \frac{\beta}{2} \sum_{|x-y|=1} s(x)s(y).$$

*Email address: ocarroll@fisica.ufmg.br

$d\mu(s)$ is a product of single spin distribution; i.e., $d\mu(s) = \Pi_x e^{-V(s(x))} ds(x)$. We only consider the case of even SSD; i.e., $V(s) = V(-s)$. $V(s)$ is bounded from below and increases at infinity at least quadratically. Expectations of the probability measure $e^{S(s)} d\mu(s)$/normalization are denoted by $\langle\ \rangle$. Truncated CF's are given by derivatives with respect to J of $\ln Z(J)$ at $J=0$. The above is formal and we actually start with the system on a finite lattice. Then for sufficiently small β the thermodynamic limit as well as exponential tree decay of the truncated CF's is established using the polymer expansion [1,4].

We now describe the organization of the paper. In Sec. II we introduce the associated lattice quantum field theory (QFT) and give the Feynman-Kac (FK) formula that relates the CF of the spin system to the vacuum expectation values of the QFT. In particular, a spectral representation of a partially truncated four-point function is obtained, which is the object of our analysis for the detection of bound states. The BS equation for the four-point function is introduced in Sec. III and using it we obtain our results. The generalization to N-component vector models is given in Sec. IV. We make some concluding remarks in Sec. V.

II. FEYNMAN-KAC FORMULA

In this section we introduce the associated lattice quantum field theory and establish a FK formula. Taking the infinite lattice limit in the x_o direction (called time), the associated lattice quantum field theory can be constructed in a standard way [1,2]. The construction provides the quantum mechanical Hilbert space $\mathcal{H}$ with inner product (,), commuting self-adjoint energy-momentum operators $H \geqslant 0$, $\vec{P}$, the time zero field operator $\hat{s}(\vec{x})$, $x=(0,x)$, and the vacuum vector Ω. The relation of the Hilbert space objects to the CF's is given by the Feynman-Kac formula, denoting $\hat{s}(0)=\hat{s}$, $x_k=(t_k,\vec{x}_k)$, and for $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$,

$$[\Omega, f_1(\hat{s}) e^{-H(t_2-t_1)} e^{i\vec{P}(\vec{x}_1-\vec{x}_2)} f_2(\hat{s}) e^{-H(t_3-t_2)} \times e^{i\vec{P}(\vec{x}_2-\vec{x}_3)} \cdots f_n(\hat{s})\Omega] = \langle f_1(s(x_1)) \dots f_n(s(x_n))\rangle,$$

where f_i are functions of the time zero spin configurations.

We now state some known results on the spectrum that we need here. The one-particle states are generated by vectors of the form $\hat{s}(\vec{x})\Omega$ and by the methods of [4], have mass $m \sim -\ln\beta$ for β small, and have an isolated real analytic dispersion curve $w(\vec{p}) \geqslant w(0) \equiv m$. The EM dispersion curve is determined as the zero of $\tilde{\Gamma}(p_o = iw(\vec{p}),\vec{p})$, where $\tilde{\Gamma}(p)$ is the Fourier transform of $\Gamma(x,y)$. $\Gamma(x,y)$ is minus the convolution inverse of the two-point function $\langle s(x)s(y)\rangle = S(x,y)$. To lowest order in β,

$$w(\vec{p}) = -\ln\beta - \ln\langle s^2\rangle - 2\beta\langle s^2\rangle + \beta\langle s^2\rangle 2\sum_{i=1}^{d-1}(1-\cos p_i).$$

Furthermore, there is no EM spectrum up to $-(2-\varepsilon)\ln\beta$, $\varepsilon(\beta) > 0$, and $\varepsilon(\beta)\downarrow 0$ as $\beta\downarrow 0$. This is known as the upper mass gap property, and the Orstein-Zernicke behavior of the two-point function is a consequence [3]. The general representation for $\tilde{S}(p)$, the Fourier transform of S, can be obtained by adapting the work of [2,4] to show that

$$\tilde{S}(p_o,\vec{p}) = \frac{\sinh w(\vec{p}) Z(\vec{p})(2\pi)^{d-1}}{\cosh w(\vec{p}) - \cos p_o} + \int_{\cosh \bar{m}}^{\infty} \frac{d\eta(a,\vec{p})}{a - \cos p_o},$$

where $\bar{m} = -(3-\varepsilon')\ln\beta$, $\varepsilon' > 0$, $d\eta$ is a positive measure and $Z(p)$, $w(\vec{p})$ are real analytic in p; $\bar{m}$ is a lower bound for the onset of the three-particle spectrum.

To determine the mass spectrum (EM spectrum at $\vec{p}=0$) in the interval $(m,2m)$, we consider the states in the subspace generated by $\hat{s}(\vec{x})\hat{s}(\vec{y})\Omega$. The truncated four-point function related to this state (after subtracting out the vacuum contribution) is

$$D(x_1x_2;x_3x_4) = \langle s(x_1)s(x_2)s(x_3)s(x_4)\rangle - \langle s(x_1)s(x_2)\rangle\langle s(x_3)s(x_4)\rangle,$$

where $x_i = (t_i,\vec{x}_i)$. By translation, invariance D depends only on the difference variables. We now introduce the newly devised relative coordinates (ξ,η,τ), which are the substitute for the center of mass and relative coordinates used in the continuum [8]. Let $\xi = x_2 - x_1$, $\eta = x_4 - x_3$, $\tau = x_3 - x_2$, and let us denote by p, q, k the respective Fourier transform variables. Writing $\xi = (\xi_o,\xi)$, etc., it follows that if $\xi_o = \eta_o = 0$, $D(\xi,\eta,\tau) = [\theta(-\xi), e^{-H|\tau|} e^{i\vec{p}\cdot\vec{\tau}}\theta(\vec{\eta})]$, where $\theta(\vec{\eta}) = \hat{s}(\vec{o})\hat{s}(\vec{\eta})\Omega - [\Omega,\hat{s}(\vec{o})\hat{s}(\vec{\eta})\Omega]\Omega$. A calculation shows, with $f: Z^d \to C$, a function of space position only, and letting $\tilde{f}(\vec{p})$ and $\tilde{D}(p,q,k)$ denote the Fourier transform of f and D,

$$\int d^d p\, d^d q\, \bar{\tilde{f}}(\vec{p})\tilde{f}(\vec{q})D(p,q,k) = \int_0^\infty \int_{T_{d-1}} \frac{\sinh E}{\cosh E - \cos k_o}(2\pi)^{3d+2} \times \delta(\vec{q}-\vec{k})\, d[\theta(f), \mathcal{E}(E,\vec{q})\theta(f)],$$

where $\mathcal{E}(E,\vec{q})$ is the spectral family associated with H, $\vec{P}$ and T^{d-1} is the d one-dimensional torus,

$$\theta(f) = \sum_{\vec{x}} f(\vec{x})\theta(-\vec{x}), \quad \vec{x} \in Z^{d-1}.$$

The singularities in k_o, for $\vec{k}$ fixed, of the left side, are points in the EM spectrum that result from considering the right side.

III. BETHE-SALPETER EQUATION

To determine these singularities below the two-particle threshold, we now consider the BS equation, which in operator form is $D = D_o + D_o K D$, and in terms of kernels is

$$D(x_1x_2;x_3x_4) = D_o(x_1x_2;x_3x_4) + \int d_{y1}d_{y2}d_{y3}d_{y4} \cdot D_o(x_1x_2;y_1y_2) \times K(y_1y_2;y_3y_4) D(y_3,y_4;x_3x_4),$$

$$D_o(x_1x_2;x_3x_4)=\langle s(x_1)s(x_2)\rangle\langle s(x_3)s(x_4)\rangle + \langle s(x_1)s(x_3)\rangle\langle s(x_2)s(x_4)\rangle,$$

where we use an integral notation for sums over lattice points. K is called the kernel of the BS equation. Using the $x_1\leftrightarrow x_2$ and $x_3\leftrightarrow x_4$ symmetry properties of D and D_o and thus also of K, the Fourier transform of the BS equation can be written in the convolution form

$$\tilde{D}(p,q,k)=\tilde{D}_o(p,p',k) + \int \tilde{D}_o(p,q',k)\tilde{K}(p',q',k)\tilde{D}(q',q,k)dp'dq',$$

or in the operator form $\tilde{D}=\tilde{D}_o+\tilde{D}_o\tilde{K}\tilde{D}$, where the action of an operator $\tilde{M}(p,q,k)$ on a function $\tilde{f}$ is given by

$$\tilde{M}\tilde{f}(p)=\int \tilde{M}(p,q,k)\tilde{f}(q)dq.$$

As

$$\tilde{D}_o(p,q,k)=\tilde{S}(p)\tilde{S}(q)\delta(-k+p+q) + \tilde{S}(p)\tilde{S}(k-p)\delta(-q+p),$$

the action of $\tilde{D}_o(k_o)\equiv\tilde{D}_o\,[k=(k_o,\vec{k}=0)]$, etc., on functions depending only on $\vec{p}$ is $\tilde{D}_o(k_o)\tilde{f}(p)=(2\pi)^{d+1}\tilde{S}(p)\tilde{S}(k-p)[f(\vec{p})+f(-\vec{p})]$, where

$$\tilde{S}(p)=\sum_x e^{-ipx}\langle s(0)s(x)\rangle.$$

Now we can write $\tilde{D}=\tilde{D}_o(1-\tilde{K}\tilde{D}_o)^{-1}$ in the form, for $f(\vec{p})=f(-\vec{p})$,

$$[f,\tilde{D}(k_o)f]=2\int d^dp\tilde{f}(\vec{p})G(\vec{p},k_o)\tilde{g}(\vec{p},k_o),$$

where

$$G(\vec{q},k_o)=\int dq_o\tilde{S}(q)\tilde{S}(k_o-q_o,\vec{q})$$

and

$$\tilde{g}(\cdot,k_o)=[1-(2\pi)^{-2(d+1)}\tilde{K}(k_o)\tilde{D}(k_o)]^{-1}\tilde{f}.$$

As $G(\vec{q},k_o)$ is analytic in $|\mathrm{Im}\,k_o|<2m$, the only singularities in $|\mathrm{Im}\,k_o|<2m$ come from $g(\cdot,k_o)$ which in turn come from $\mathrm{Im}\,k_o$ where the inverse of $1-(2\pi)^{-2(d+1)}\tilde{K}(k_o)\tilde{D}(k_o)$ does not exist.

Our approximation (called the ladder approximation) replaces $\tilde{K}(k_o)$ by $\tilde{L}(k_o)$, the leading term in an expansion in β for $\tilde{K}(k_o)$, which turns out to be local in space time and independent of β. We obtain

$$\tilde{L}(p,q,k)=\frac{1}{2s\langle s^2\rangle^2}\left[\frac{\langle s^4\rangle-3\langle s^2\rangle^2}{\langle s^4\rangle-\langle s^2\rangle^2}\right]\equiv\frac{\gamma}{2\langle s^2\rangle^2}=p\,,$$

and in this approximation $[f,\tilde{D}(k_o)f]$ can be written as

$$(f,\tilde{D}(k_o)f)=\int \bar{\tilde{f}}(\vec{p})\tilde{D}_0(p,k_o)f(\vec{p})dp + \frac{\rho}{1-\rho I}\left(\int \bar{\tilde{f}}(\vec{p})\tilde{D}_o(p,k_o)dp\right) \times\left(\int \tilde{D}_o(p',k_o)f(\vec{p}\,')dp'\right),$$

where

$$\tilde{D}_o(p,k)\equiv2\tilde{S}(p)\tilde{S}(k_o-p_o,-\tilde{p})\quad\text{and}\quad I\equiv\int \tilde{D}_o(p,k_o)dp.$$

Thus we have a bound state for $\rho I=1$. Using the representation for $\tilde{S}(p_o,\vec{p})$, we can explicitly perform the p_o integration in I. Only the product of the one-particle terms can give rise to a singularity in $\mathcal{X}$ as $\mathcal{X}\uparrow 2m$, where we set $k_o=(i\chi,k=0)$. The other terms are analytic in λ up to at least $-(4-\varepsilon')\ln\beta$. Keeping only these terms, denoting the result by I_1, we obtain, writing $\mathcal{X}=zm-\varepsilon$,

$$I_1=2(2\pi)^{d-1}\int\frac{(1-e^{-4w})Z(\vec{p})^2d\vec{p}}{(1-e^{2(m-w)-\epsilon}-\varepsilon^{-2m-2w+\varepsilon}+e^{-4w})}>0$$

For $\beta=0$, using $Z(\vec{p})=\langle s^2\rangle/(2\pi)^{d-1}+0(B)$,

$$I_1=\frac{2\langle s^2\rangle^2}{l-e^{-\varepsilon}}$$

Thus, for $\alpha>0$ there is a bound state for $\gamma(1-e^{-\varepsilon_b})=1$, with mass $m_b\approx2m-\varepsilon_b=2m+\ln(1-\gamma)$. Note that for a Gaussian SSD the numerator of ρ is zero: the denominator is $\geqslant0$ by the Cauchy-Schwarz inequality. The Gaussian domination inequality of [3,6] implies $\langle s^4\rangle\leqslant3\langle s^2\rangle^2$, which implies the absence of bound states. In this case, $\rho\leqslant0$. Our analysis does not apply to the Ising case due to the zero in the denominator of c_o. Nevertheless, Gaussian domination inequalities hold for the Ising model and exclude bound states from the spectrum.

IV. *N*-COMPONENT VECTOR MODEL

We now consider N-component vector models with even rotationally invariant $[O(N)]$ SSD. The spin variable at site $x\in Z^d$ is denoted by $s(x)\in R^N$ with components $s_i(x)\in R$, $i=1,2,\ldots,N$ and the time zero operators are denoted by $\hat{s}(\vec{x}),\vec{x}\in Z^{d-1}$. The one-particle states are generated by vectors of the form

$$\hat{s}_k(\vec{x})\Omega.$$

The two-particle states are generated by

$$\hat{s}_k(\vec{x})\hat{s}_{\ell}(\vec{y})\Omega,$$

and these states can be decomposed into the rotationally invariant state

$$\hat{s}(\vec{x})\cdot\hat{s}(y)\Omega$$

and the traceless states

$$\left(\hat{s}_k(\vec{x})\hat{s}_\ell(\vec{y}) - \frac{1}{N}\hat{s}(\vec{x})\cdot\hat{s}(\vec{y})\right)\delta_{k\ell}.$$

The traceless states can further be decomposed into the symmetric and antisymmetric states

$$\left(\hat{s}_k(\vec{x})\hat{s}_\ell(\vec{y}) + \hat{s}_\ell(\vec{x})\hat{s}_k(\vec{y}) - \frac{2}{N}\hat{s}(\vec{x})\cdot\hat{s}(\vec{y})\right)\Omega$$

and

$$[\hat{S}_k(\vec{x})\hat{S}_\ell(\vec{y}) - \hat{S}_\ell(\vec{x})\hat{S}_k(\vec{y})]\Omega,$$

respectively.

We only consider the rotationally invariant state. Associated with this state (after subtracting out the vaccuum contribution) is the CF

$$D(x_1x_2x_3x_4) = \langle s(x_1)\cdot s(x_2)s(x_3)\cdot s(x_4)\rangle - \langle s(x_1)\cdot s(x_2)\rangle\langle s(x_3)\cdot s(x_4)\rangle$$

and the FK formula and spectral repetition equation remain valid. In the BS equation $D = D_o + D_oKD$, we take

$$D_o(x_1x_2x_3x_4) = \langle s(x_1)\cdot s(x_3)\rangle\langle s(x_2)\cdot(x_4)\rangle + \langle s(x_1)\cdot s(x_4)\rangle\langle s(x_2)\cdot s(x_3)\rangle.$$

As before, D and D_o are decomposed into the diagonal and nondiagonal parts, the diagonal parts being of the order β.

We find

$$\begin{aligned} D_d(x_1x_2x_3x_4) &= N[\langle s_l^4\rangle - N\langle s_l^2\rangle + (N-1)\langle s_l^2 s_2^2\rangle]\,\delta(x_3-x_1) \\ &\quad\times\delta(x_4-x_2)\,\delta(x_2-x_1) + N\langle s_1^2\rangle^2\delta(x_3-x_1) \\ &\quad\times\delta(x_4-x_2)[1-\delta(x_2-x_1)] + 0(\beta), \end{aligned}$$

$$\begin{aligned} D_{od}(x_1x_2x_3x_4) &= 2N\langle S_1^2\rangle^2\delta(x_3-x_1)\,\delta(x_4-x_2)\,\delta(x_2-x_1) \\ &\quad + N\langle S_1^2\rangle^2\delta(x_3-x_1)\,\delta(x_4-x_2)[1-\delta(x_2-x_1)] \\ &\quad + 0(\beta), \end{aligned}$$

and for $K = D_{od}^{-1} - D_d^{-1}$

$$\begin{aligned} K &= D_{od}^{-1} - D_d^{-1} + 0(\beta) \\ &= \frac{1}{N}\left[\frac{\langle s_1^4\rangle - N\langle s_1^2\rangle + (N-1)\langle s_1^2 s_2^2\rangle - 2\langle s_1^2\rangle^2}{2\langle s_1^2\rangle^2(\langle s_1^4\rangle - N\langle s_1^2\rangle^2) + (N-1)\langle s_1^2 s_2^2\rangle^2}\right] \\ &\quad\times\delta(x_3-x_1)\,\delta(x_4-x_2)\,\delta(x_2-x_1) + 0(\beta). \end{aligned}$$

Dropping the $0(\beta)$ terms, rewriting in terms of s, and taking the Fourier transform gives

$$\tilde{L}(\vec{p},\vec{q},k) = \alpha_N,$$

where α_N is given in the Introduction. The rest of the analysis goes through as in the preceding section.

V. CONCLUDING REMARKS

We have found a simple criterion for the existence of bound states based on the sign of α. The question arises as to the existence and number of bound states for large values of β in any dimension. Also, there is the question of whether or not the result generalizes to the case of noneven SSC. For example, if α is calculated using zero average fields, does the sign α still determine the presence or absence of bound states? The existence of weakly bound, bound states in lattice gauge and gauge-matter models (strongly bound, bound states are present) is also an open question.

APPENDIX: LATTICE BS EQUATION

Here we deduce a composition of kernel form for the Fourier transform of the lattice BS equation. We use the relative and conjugate variables

$$\xi = x_2 - x_1,\quad p,\quad \eta = x_4 - x_3,\quad q,\quad \tau = x_3 - x_2,\quad k,$$

and use an integral notation for lattice sums. The BS equation is

$$\begin{aligned} D(x_1x_2x_3x_4) &= D_o(x_1x_2x_3x_4) \\ &\quad + \int dy_1dy_2dy_3dy_4 D(x_1x_2y_1y_2) \\ &\quad\times K(y_1y_2y_3y_4)D_o(y_3y_4x_3x_4). \end{aligned} \tag{A1}$$

All kernels are assumed to be translationally invariant. In terms of the relative variables ξ,η,τ we write, using a bar notation for the function of the relative variables, i.e.,

$$\overline{D}(\xi,\eta,\tau) = D(0, x_2 - x_1 = \xi, x_3 - x_1 = \xi + \tau, x_4 - x_1 = \xi + \eta + \tau),$$

etc. The kernels D_1D_o and, consequently, k, are invariant under the substitutions

$$(x_1x_2x_3x_4) \to (x_2x_1x_3x_4), \tag{A2a}$$

$$(x_1x_2x_3x_4) \to (x_1x_2x_4x_3), \tag{A2b}$$

which imply

$$\overline{K}(\xi,\eta,\tau) = \overline{K}(-\xi,\eta,\tau+\xi), \tag{A3a}$$

$$\overline{K}(\xi,\eta,\tau) = \overline{K}(\xi,-\eta,\tau+\eta). \tag{A3b}$$

We introduce the variables $\xi',\eta',\tau',\ \tau''$, where

$$\xi' = y_2 - y_1, \tag{A4a}$$

$$\eta' = y_4 - y_3, \tag{A4b}$$

$$\tau' = y_1 - x_2, \tag{A4c}$$

$$\tau'' = x_3 - y_4. \tag{A4d}$$

Then

$$y_1 = \tau' + x_2, \tag{A5a}$$

$$y_2=\xi'+y_1=\xi'+\tau'+x_2, \tag{A5b}$$

$$y_4=x_3-\tau'', \tag{A5c}$$

$$y_3=y_4-\eta'=x_3-\tau'''-\eta'. \tag{A5d}$$

We have

$$D(x_1x_2y_1y_2)=\overline{D}(\xi,\xi',\tau'), \tag{A6a}$$

$$D_0(y_3y_4x_3x_4)=\overline{D}_0(\eta',\eta,\tau''), \tag{A6b}$$

$$\begin{aligned}\overline{K}(y_1y_2y_3y_4)&=\overline{K}(\xi',\eta',\tau-\tau'-\tau''-\xi'-\eta')\\&=\overline{K}(-\xi',-\eta',\tau-\tau'-\tau''),\end{aligned} \tag{A6c}$$

where for the first equality of Eq. (A6c) we use Eq. (A5), i.e.,

$$\begin{aligned}y_3-y_2&=(x_3-\tau''-\eta')-(\xi'+\tau'+x_2)\\&=\tau-\tau'-\tau''-\xi'-\eta',\end{aligned}$$

and for the second we use Eq. (A3).

Thus the BS equation becomes

$$\overline{D}(\xi,\eta,\tau)=\overline{D}_0(\xi,\eta,\tau)+\int d\xi'd\eta'd\tau'd\tau''\overline{D}(\xi,\xi',\tau'),$$

$$\overline{K}(-\xi',-\eta',\tau-\tau'-\tau'')\overline{D}_0(\eta',\eta,\tau''). \tag{A7}$$

The way that the variables enter in Eq. (A7) is important, as taking the Fourier transform of Eq. (A7) with the conjugate variables p,q,k and dropping the bar gives the desired form

$$\tilde{D}(p,q,k)=\tilde{D}_0(p,q,k)-\frac{1}{(2\pi)^{2d}}\times\int dp'dq'\tilde{D}(p,p',k)K(p',q',k)\tilde{D}_0(q',q,k). \tag{A8}$$

[1] J. Glimm and A. Jaffe, *Quantum Physics*, 2nd ed. (Springer, New York, 1986).

[2] R. Schor, Commun. Math. Phys. **59**, 213 (1978).

[3] P. Paes-Leme, Ann. Phys. (N.Y.) **115**, 367 (1978).

[4] B. Simon, *Statistical Mechanics of Lattice Models* (Princeton University Press, Princeton, NJ, 1994).

[5] T. Spencer and F. Zirilli, Commun. Math. Phys. **49**, 1 (1976).

[6] J. Dimock and J. P. Eckman, Commun. Math. Phys. **51**, 41 (1976).

[7] D. Brydges, J. Fröhlich, and T. Spencer, Commun. Math. Phys. **83**, 123 (1983).

[8] R. Schor, J. Barata, P. Veiga, and E. Pereira, Phys. Rev. E **59**, 2689 (1999).